



A Survey of the Earth Dams by Sturm-Liouville Equations - The Singular Case

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Abstract

In this work, we investigate the vibrations of embankments by the singular Sturm-Liouville equations. At first, we create the mathematical form of the vibrations by the shear beam (SB) model (see [21]) and transform this given form to the Sturm-Liouville form with a singularity. Finally, we discuss the numerical solution to the considered problem using the variational iteration method.

Keywords: Sturm-Liouville equation; singularity; embankment.

1 Introduction

The earth dams that have been built by compact soils are vulnerable in displacements. The scientists can find the steady of embankments by studding and analyzing of their seismic response. These investigations and their findings can help us to know the resistance of dams during an earthquake. In the making of the mathematical model, the formulation is derived from a two-dimensional structure within a fixed domain. The obtained formulae are applicable in various places. The vibrations of embankments usually led to differential equations with boundary conditions. In [7, 21], these problems were studied without using the Sturm-Liouville problems which the specific shear strain of the soil in the body of a dam has been taken. We used this shear strain and created the mathematical model of the problem by Sturm-Liouville equations (see [18, 19]). In [18], taking the shear stress $\tau_{yz} = G \frac{\partial u}{\partial z}$, we gave the following differential equation for $\alpha = \pi \left(H^{-\frac{m}{2}+1} - h^{-\frac{m}{2}+1} \right)^{-1}$,

$$\frac{d^2y}{ds^2} + \left(\lambda^2 - \frac{\frac{m^2-4}{16} \alpha^{-\frac{2}{1-\frac{m}{2}}}}{\left(s + \alpha h^{-\frac{m}{2}+1} \right)^{\frac{2}{1-\frac{m}{2}}}} \right) y = 0, \quad 0 \leq s \leq \pi,$$

where m is a parameter in interval $[0, 1]$. The scholars are focused on the shear stress-strain of the soil in theses problems. In this article, we take a new type of the shear strain that gives a new potential in Sturm-Liouville equations the so-called potential with singularities. Applying the shear beam (SB) model, we will make a partial differential equation and the separation variant transforms it to the ordinary differential equation. Then by a transformation, we get a suitable Sturm-Liouville equation of this problem. We will take the singular Sturm-Liouville equations in the survey of vibrations of dams which this technique is a new method in this field. In the other hand, the Sturm-Liouville problems were studied in many works in the last decades (see [2, 3, 9, 10, 11, 16, 17, 20, 22]).

There exist various methods to solve the differential equation such as variational iteration method, optimal perturbation iteration method, perturbation iteration method, Taylor collocation method, etc (see [4, 5, 6, 8]). The variational iteration method which is an appropriate analytical method to solve the linear or nonlinear problems was taken by researchers to survey the Sturm-Liouville equations (see [1]). This method was introduced by He and many scholars have used it to get the solution of the differential equations. For example, Bildik and Deniz have applied this method to systems of delay differential equations with initial conditions (see [6]). To complete our mathematical analysis, we will take the variational iteration method for the singular Sturm-Liouville equation in our work. By studding this article, we can find that the used technique is an effective way in the investigation of various classes of such applied problems.

Taking the physical rules, we can outline a following form for the vibrations of embankments with the shear stress $\tau_{yz} = G \left(\frac{\partial u}{\partial z} + \frac{u}{z-H} \right)$ as

$$\frac{1}{z} \frac{d}{dz} \left(z \left(\frac{d\vartheta}{dz} + \frac{\vartheta}{z-H} \right) \right) = -\lambda\vartheta, \quad h \leq z < H, \tag{*}$$

$$U(\vartheta) := \vartheta'(h) + \frac{1}{h-H} \vartheta(h) = 0, \quad V(\vartheta) := \vartheta(H) = 0,$$

wherein λ is a spectral parameter. Also h and H are real constants.

The rest of this paper is organized as follows. In Section 2, we present the mathematical form of the problem and then we transform this model to a suitable Sturm-Liouville equation. Section 3 contains some numerical statements of the considered problem that we have applied the variational iteration method for the singular differential equation with the boundary condition. The conclusion is given in Section 4.

2 Formulation of the Model

This section contains some corollaries which clarify the main goal of this article. We establish a mathematical form of vibrations of dams by taking the shear beam technique.

The soil elements in dams have a volume

$$V(z) = b(z)Ldz, \quad h \leq z < H,$$

where $b(z) = \frac{B}{H}z$. Therefore

$$V(z) = \frac{BL}{H}zdz. \tag{1}$$

Let F_H be a shear force on the horizontal surface that a vibration will exert. We can write

$$F_H = \tau_{yz}A_{xy}(z),$$

where τ_{yz} and $A_{xy}(z)$ are the shear stress and the area in the xy -plane, respectively. Assume that $G = G_b$ be the soil average shear modulus and $u(z; t)$ be the displacement at z . Define

$$A_{xy}(z) = b(z)L, \rightarrow A_{xy}(z) = \frac{B}{H}zL,$$

$$\tau_{yz} = G \left(\frac{\partial u}{\partial z} + \frac{u}{z - H} \right).$$

We will have

$$dF_H = -\frac{BL}{H} \frac{\partial(z\tau_{yz})}{\partial z} dz.$$

Therefore

$$dF_H = -\frac{BLG_b}{H} \frac{\partial}{\partial z} \left(z \left(\frac{\partial u}{\partial z} + \frac{u}{z - H} \right) \right) dz. \tag{2}$$

Besides, the sum total of inertial forces for elements is

$$I = \rho_s V(z)\ddot{u},$$

wherein $\rho_s :=$ soil mass density in dams, and $\ddot{u} :=$ acceleration. Using (1), we get

$$I = \rho_s \frac{BL}{H} z dz \frac{\partial^2 u}{\partial t^2}. \tag{3}$$

The static of the elements is satisfied when the net force dF_H equals to the inertia forces I . These conclude that

$$dF_H + I = 0. \tag{4}$$

Substituting (2) and (3) in (4), the motion equation

$$\frac{BLG_b}{H} \frac{\partial}{\partial z} \left(z \left(\frac{\partial u}{\partial z} + \frac{u}{z-H} \right) \right) dz = \rho_s \frac{BL}{H} z dz \frac{\partial^2 u}{\partial t^2},$$

is achieved. By simplifying this equation, we have the following PDE

$$\frac{1}{z} \frac{\partial}{\partial z} \left(z \left(\frac{\partial u}{\partial z} + \frac{u}{z-H} \right) \right) = \frac{\rho_s}{G_b} \frac{\partial^2 u}{\partial t^2}. \tag{5}$$

Then applying (5) and considering $C_b^2 = \frac{\rho_s}{G_b}$, the equation

$$\frac{1}{z} \frac{\partial}{\partial z} \left(z \left(\frac{\partial u}{\partial z} + \frac{u}{z-H} \right) \right) = C_b^2 \frac{\partial^2 u}{\partial t^2}, \tag{6}$$

will be given. If we consider

$$u(z; t) = \vartheta(z)\theta(t), \tag{7}$$

and replace in (6), the ordinary differential equation of this problem is given as follows

$$\frac{1}{z} \frac{d}{dz} \left(z \left(\frac{d\vartheta}{dz} + \frac{\vartheta}{z-H} \right) \right) = -\lambda\vartheta, \tag{8}$$

where λ is a spectral parameter. Therefore we can write

$$\frac{d^2\vartheta}{dz^2} + \left(\frac{1}{z} + \frac{1}{z-H} \right) \frac{d\vartheta}{dz} + \left(\frac{1}{z(z-H)} - \frac{1}{(z-H)^2} \right) \vartheta = -\lambda\vartheta. \tag{9}$$

Here, the boundary conditions are in two cases:

- (1) : Since the embankment has stuck in the base, we have $u(H; t) = 0$,
- (2) : Since there is not any stress in the top surface, we have $\tau_{yz}(h; t) = 0$.

Corollary 1. We can consider the following BVP(L) with $\tau_{yz} = G \left(\frac{\partial u}{\partial z} + \frac{u}{z-H} \right)$,

$$\frac{d^2\vartheta}{dz^2} + \left(\frac{1}{z} + \frac{1}{z-H} \right) \frac{d\vartheta}{dz} + \left(\frac{1}{z(z-H)} - \frac{1}{(z-H)^2} \right) \vartheta = -\lambda\vartheta, \quad h \leq z < H, \tag{10}$$

$$U(\vartheta) := \vartheta'(h) + \frac{1}{h-H}\vartheta(h) = 0, \quad V(\vartheta) := \vartheta(H) = 0. \tag{11}$$

Now we transform (10) to the Sturm-Liouville equation. These equations were studied in [3, 9, 15, 16, 23]. For this purpose, we take the following transformation [12].

Lemma 2. The following transformation

$$y = \exp \left(\frac{1}{2} \int_a^z p(s) ds \right) \vartheta, \tag{12}$$

transforms the ODE

$$\vartheta''(z) + p(z)\vartheta'(z) + q(z)\vartheta(z) = 0, \quad z \in [a, b], \tag{13}$$

to

$$\frac{d^2y}{dz^2} + \left(q(z) - \frac{1}{4}p^2(z) - \frac{1}{2}p'(z) \right) y = 0. \tag{14}$$

Proof. Considering (12), we have

$$\vartheta(z) = \exp\left(-\frac{1}{2} \int_a^z p(s) ds\right) y(z).$$

The first and second order derivatives of ϑ can be given

$$\begin{aligned} \vartheta'(z) &= \exp\left(-\frac{1}{2} \int_a^z p(s) ds\right) \left(y'(z) - \frac{1}{2} p(z) y(z)\right), \\ \vartheta''(z) &= \exp\left(-\frac{1}{2} \int_a^z p(s) ds\right) \left(y''(z) - p(z) y'(z) + \left(\frac{1}{4} p^2(z) - \frac{1}{2} p'(z)\right) y(z)\right). \end{aligned}$$

Now substituting these derivatives in (13), we get

$$\begin{aligned} &\left(y''(z) - p(z) y'(z) + \left(\frac{1}{4} p^2(z) - \frac{1}{2} p'(z)\right) y(z)\right) \exp\left(-\frac{1}{2} \int_a^z p(s) ds\right) \\ &+ p(z) \left(y'(z) - \frac{1}{2} p(z) y(z)\right) \exp\left(-\frac{1}{2} \int_a^z p(s) ds\right) + q(z) y(z) \exp\left(-\frac{1}{2} \int_a^z p(s) ds\right) = 0. \end{aligned}$$

Therefore

$$y''(z) - p(z) y'(z) + \frac{1}{4} p^2(z) y(z) - \frac{1}{2} p'(z) y(z) + p(z) y'(z) - \frac{1}{2} p^2(z) y(z) + q(z) y(z) = 0,$$

and finally

$$y''(z) + \left(q(z) - \frac{1}{4} p^2(z) - \frac{1}{2} p'(z)\right) y(z) = 0.$$

The proof is completed. □

Now taking this transformation, the equation (10) turns to

$$-y'' + \frac{4Hz - H^2}{(z - H)^2} y = \lambda y, \quad h \leq z < H, \tag{15}$$

the so-called the Sturm-Liouville equation with a singularity.

Corollary 3. The motion equation of vibrations of embankments with the special shear stress

$\tau_{yz} = G \left(\frac{\partial u}{\partial z} + \frac{u}{z-H}\right)$ is as follows

$$-y'' + \frac{q_0(z)}{(z - H)^2} y = \lambda y, \quad h \leq z < H, \tag{16}$$

where $q_0(z) = \frac{4Hz - H^2}{4z^2}$. Also boundary conditions are

$$U(y) := y'(h) - \beta y(h) = 0, \quad V(y) := y(H) = 0, \tag{17}$$

where $\beta = \frac{H}{2h(H-h)}$. So we can write a boundary value problem (16)-(17) as the problem of the seismic response of earth dams with this stress.

3 Application of VIM

In this section, we establish some numerical consequences and utilize the variational iteration method (VIM) to solve the problem (16)-(17) which is a Sturm-Liouville problem (see [1, 6, 13, 14, 22]). It is well known that the variational iteration method is an effective tool to solve a wide class of linear and nonlinear problems with a fast convergence to exact solutions.

To demonstrate the technique, we assume that

$$L[u(t)] + N[u(t)] = g(t),$$

be a differential equation with the linear operator L , nonlinear operator N and continuous function $g(t)$.

We establish a correction function which is the basic character of VIM as

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^t \mu \{Lu_n(s) + N\tilde{u}_n(s) - g(s)\} ds,$$

where μ is a general Lagrange multiplier that it can be identified optimally by variational theory, u_n is the n th approximate solution, and \tilde{u}_n denotes a restricted variation, i.e., $\delta\tilde{u}_n = 0$. In this way, we can obtain an exact solution for linear problems only by one iteration step as regards the Lagrange multiplier is exactly identified.

Now we would like to solve the differential equation which arises from the displacement of the embankment in the special case by VIM. So we consider the following differential equation

$$-y'' + \frac{\nu}{(z - H)^2}y = \lambda y, \quad h \leq z < H, \tag{18}$$

for a real parameter ν . In the case $\nu = 0$, this equation together with initial conditions $y(h) = 1$ and $y'(h) = \beta$, for $\beta = i\sqrt{\lambda}$ has the solution $e(z) = \exp(i\sqrt{\lambda}(z - h))$. To find the approximate analytical solution by using this method, we have the correction functional

$$y_{n+1}(z) = y_n(z) + \int_h^z \mu \left\{ \frac{d^2 y_n(s)}{ds^2} - \frac{\nu}{(s - H)^2} \tilde{y}_n(s) + \lambda y_n(s) \right\} ds, \tag{19}$$

where \tilde{y}_n is assumed as a restricted variation. In the following, by making the functional stationary

$$\begin{aligned} \delta y_{n+1}(z) &= \delta y_n(z) + \delta \int_h^z \mu \left\{ \frac{d^2 y_n(s)}{ds^2} - \frac{\nu}{(s - H)^2} \tilde{y}_n(s) + \lambda y_n(s) \right\} ds \\ &= \delta y_n(z) + \mu(s) \delta y'_n(s)|_{s=z} - \mu'(s) \delta y(s)|_{s=z} \\ &\quad + \int_h^z \left\{ \frac{d^2 \mu(s)}{ds^2} + \lambda \mu(s) \right\} \delta y_n(s) ds \\ &= (1 - \mu'(z)) \delta y_n(z) + \mu(z) \delta y'_n(z) \\ &\quad + \int_h^z \left\{ \frac{d^2 \mu(s)}{ds^2} + \lambda \mu(s) \right\} \delta y_n(s) ds, \end{aligned}$$

the stationary conditions can be obtained

$$\begin{cases} \frac{d^2 \mu(s)}{ds^2} + \lambda \mu(s) = 0, \\ \mu(z) = 0, \quad \mu'(z) = 1. \end{cases} \tag{20}$$

We give the Lagrange multiplier $\mu = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(z - s))$. So we take the VIM iteration formula

$$y_{n+1}(z) = y_n(z) + \int_h^z \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(z - s)) \left\{ y_n''(s) - \frac{\nu}{(s - H)^2} y_n(s) + \lambda y_n(s) \right\} ds. \tag{21}$$

Substituting the initial approximation $y_0(z, \lambda) = \exp(i\sqrt{\lambda}(z - h))$ in (21), we have

$$y_1(z) = \exp(i\sqrt{\lambda}(z - h)) - \int_h^z \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(z - s)) \left\{ \frac{\nu}{(s - H)^2} \exp(i\sqrt{\lambda}(s - h)) \right\} ds. \tag{22}$$

By using Mathematica, the following approximation of the solution is given

$$y(z) \approx y_1(z) = \exp(i\sqrt{\lambda}(z - h)) + \frac{\nu}{\sqrt{\lambda}} \left(\frac{\sin(\sqrt{\lambda}(z - h))}{(H - h)^2} + \frac{2\sin(\sqrt{\lambda}(\frac{z}{2} - \frac{h}{2})) \exp(i\sqrt{\lambda}(\frac{z}{2} - \frac{h}{2}))}{(\frac{h}{2} - H + \frac{z}{2})^2} + \frac{4\sin(\sqrt{\lambda}(\frac{z}{4} - \frac{h}{4})) \exp(i\sqrt{\lambda}(\frac{3z}{4} - \frac{3h}{4}))}{(\frac{h}{4} - H + \frac{3z}{4})^2} + \frac{4\sin(\sqrt{\lambda}(\frac{3z}{4} - \frac{3h}{4})) \exp(i\sqrt{\lambda}(\frac{z}{4} - \frac{h}{4}))}{(\frac{3h}{4} - H + \frac{z}{4})^2} \right).$$

This approximate solution is shown in Fig. 1 for $h = 10, H = 100, \lambda = 4$ and $\nu = 1$.

Here we want to use the suggested method again to survey the problem arisen from the motion of the embankment. To do this, consider the differential equation as follow

$$-y'' + q(z)y = \lambda y, \quad h \leq z < H, \tag{23}$$

where $q(z) = \frac{4Hz - H^2}{4z^2(z - H)^2}$. When $q(z) = 0$, the solution of (23) subject to the initial conditions $y(h) = 1$ and $y'(h) = \beta$, for $\beta = i\sqrt{\lambda}$ is $e(z) = \exp(i\sqrt{\lambda}(z - h))$. We take the correction functional

$$y_{n+1}(z) = y_n(z) + \int_h^z \mu \left\{ \frac{d^2 y_n(s)}{ds^2} - \frac{4Hs - H^2}{4s^2(s - H)^2} \tilde{y}_n(s) + \lambda y_n(s) \right\} ds, \tag{24}$$

where the restricted variation is assumed by \tilde{y}_n . By making the functional stationary

$$\begin{aligned} \delta y_{n+1}(z) &= \delta y_n(z) + \delta \int_h^z \mu \left\{ \frac{d^2 y_n(s)}{ds^2} - \frac{4Hs - H^2}{4s^2(s - H)^2} \tilde{y}_n(s) + \lambda y_n(s) \right\} ds \\ &= \delta y_n(z) + \mu(s) \delta y'_n(s)|_{s=z} - \mu'(s) \delta y(s)|_{s=z} \\ &\quad + \int_h^z \left\{ \frac{d^2 \mu(s)}{ds^2} + \lambda \mu(s) \right\} \delta y_n(s) ds \\ &= (1 - \mu'(z)) \delta y_n(z) + \mu(z) \delta y'_n(z) \\ &\quad + \int_h^z \left\{ \frac{d^2 \mu(s)}{ds^2} + \lambda \mu(s) \right\} \delta y_n(s) ds, \end{aligned}$$

we have (20). We can give the Lagrange multiplier $\mu = \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(z - s))$. So the iteration formula can be obtained as

$$\begin{aligned} y_{n+1}(z) &= y_n(z) \\ &\quad + \int_h^z \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(z - s)) \left\{ y_n''(s) - \frac{4Hs - H^2}{4s^2(s - H)^2} y_n(s) + \lambda y_n(s) \right\} ds. \end{aligned} \tag{25}$$

Considering the initial approximation $y_0(z, \lambda) = \exp(i\sqrt{\lambda}(z - h))$ and substituting in (25), we can write

$$\begin{aligned} y_1(z) &= \exp(i\sqrt{\lambda}(z - h)) \\ &\quad - \int_h^z \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}(z - s)) \left\{ \frac{4Hs - H^2}{4s^2(s - H)^2} \exp(i\sqrt{\lambda}(s - h)) \right\} ds. \end{aligned} \tag{26}$$

By using again Mathematica, the following approximation of the solution is obtained

$$\begin{aligned} y(z) \approx y_1(z) &= \exp(i\sqrt{\lambda}(z - h)) \\ &\quad + \frac{1}{\sqrt{\lambda}} \left(\frac{(4hH - H^2) \sin(\sqrt{\lambda}(z - h))}{4h^2(H - h)^2} \right. \\ &\quad + \frac{(4H(\frac{h}{2} + \frac{z}{2}) - H^2) \sin(\sqrt{\lambda}(\frac{z}{2} - \frac{h}{2})) \exp(i\sqrt{\lambda}(\frac{z}{2} - \frac{h}{2}))}{2(\frac{h}{2} + \frac{z}{2})^2 (\frac{h}{2} - H + \frac{z}{2})^2} \\ &\quad + \frac{(4H(\frac{h}{4} + \frac{3z}{4}) - H^2) \sin(\sqrt{\lambda}(\frac{z}{4} - \frac{h}{4})) \exp(i\sqrt{\lambda}(\frac{3z}{4} - \frac{3h}{4}))}{(\frac{h}{4} + \frac{3z}{4})^2 (\frac{h}{4} - H + \frac{3z}{4})^2} \\ &\quad \left. + \frac{(4H(\frac{3h}{4} + \frac{z}{4}) - H^2) \sin(\sqrt{\lambda}(\frac{3z}{4} - \frac{3h}{4})) \exp(i\sqrt{\lambda}(\frac{z}{4} - \frac{h}{4}))}{(\frac{3h}{4} + \frac{z}{4})^2 (\frac{3h}{4} - H + \frac{z}{4})^2} \right). \end{aligned}$$

This approximate solution is shown in Fig. 2 for $h = 10, H = 100, \lambda = 4$.

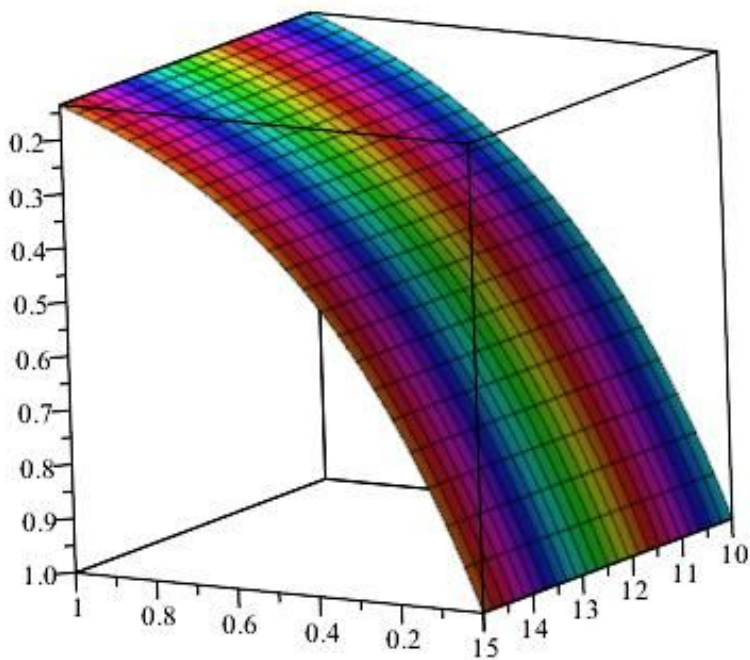


Figure 1: The approximate solution using $h = 10$, $H = 100$, $\lambda = 4$.

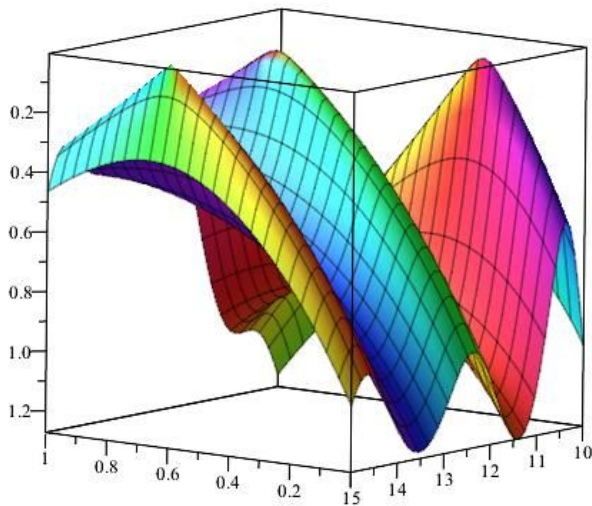


Figure 2: The approximate solution using $h = 10$, $H = 100$, $\lambda = 4$.

4 Conclusion

In this paper, the seismic response of earth dams has been studied by the Sturm-Liouville equations. The approximate solution of this problem has been computed with the use of the variational iteration method. We took the Mathematica software to compute this solution with high accuracy. The more accurate solutions help us to know the rate of vibrations of embankments more exact.

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Conflicts of Interest The authors declare no conflict of interest.

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